On the electrostatic field of a tokamak in the limit of large aspect ratio and concentric circular flux surfaces

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Abstract

From a common expression for the poloidal electrostatic field of a tokamak, in the limit of large aspect ratio and concentric circular flux surfaces, one may determine the associated potential. This potential satisfies Poisson's equation, which reduces to Laplace's equation when the medium has vanishing charge density, in axial geometry but not toroidal geometry. A simple transformation takes the potential over to the correct harmonic form for tokamak coordinates, and the resulting electrostatic field is calculated. From the radial field one may estimate the supporting charge density on the boundary, and from the poloidal field one may determine a prediction for the radial dependence of the electron temperature, which does not compare well with a rough estimate of the profile often seen in a tokamak.

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I. INTRODUCTION

From a common expression for the poloidal electrostatic field of a tokamak, in the limit of large aspect ratio and concentric circular flux surfaces, one may determine the associated potential. This potential satisfies Poisson's equation, which reduces to Laplace's equation when the medium has vanishing charge density, in axial geometry but not toroidal geometry. A simple transformation takes the potential over to the correct harmonic form for tokamak coordinates, and the resulting electrostatic field is calculated. From the radial field one may estimate the supporting charge density on the boundary, and from the poloidal field one may determine a prediction for the radial dependence of the electron temperature, which does not compare well with a rough estimate of the profile often seen in a tokamak.

II. DERIVATION

We start with the expression given by Wesson [1] for the poloidal electrostatic field at equilibrium

$$E_\theta = \frac{\langle E_\phi B_\phi / B_\theta \rangle B^2}{\langle B^2/B_\theta \rangle B_\theta} - \frac{E_\phi B_\phi}{B_\theta} + R B_\phi p' \eta_\parallel \left(\frac{\langle 1/B_\theta \rangle B^2}{\langle B^2/B_\theta \rangle B_\theta} - \frac{1}{B_\theta} \right) , \tag{1}$$

where $\langle A \rangle$ is the flux surface average of A. In the limit of large aspect ratio $\varepsilon \ll 1$ and concentric circular flux surfaces $\Delta_{GS}(r) \equiv 0$, and assuming toroidal symmetry $\partial/\partial\phi \equiv 0$, one may write $E_\theta = E_\theta^{(1)}(1 + \varepsilon \cos\theta) - E_\theta^{(2)}/(1 + \varepsilon\cos\theta)$, which reduces to [2]

$$E_\theta = E_\theta^{(2)} (4\varepsilon \cos\theta - \varepsilon^2 \cos 2\theta)/2 , \tag{2}$$

upon application of Stokes' theorem to Faraday's law [3]. The associated potential may be found from $E_\theta \equiv -\partial\Phi/r\partial\theta$, which evaluates to $\Phi_{axi} = \Phi_0(8\varepsilon\sin\theta - \varepsilon^2 \sin 2\theta)/4$. Note that this Φ_0 is *not* the Φ^0 of the GT-FRC model [4, 5] but is a unit bearing constant which sets the scale. The reason for the subscript "axi" for "axial" becomes apparent when one notices that the potential is of the correct harmonic form [6, 7] for an unweighted circle representing a cylindrical flux surface in (r, θ, z) coordinates with axial symmetry, as is easily verified in $(Z = -r\sin\theta, R = R_0 + r\cos\theta, z)$ coordinates via application of the axial Laplacian $\nabla^2_{axi} \equiv \partial^2/\partial Z^2 + \partial^2/\partial R^2$ to $\Phi_{axi} = \Phi_0 Z[(R - R_0)/2R_0^2 - 2/R_0]$. In order to achieve the correct harmonic form for tokamak geometry, the potential must satisfy the toroidal Laplacian, which in cylindrical coordinates [3, 8, 9] is given by $\nabla^2_{tor} \equiv \nabla^2_{axi} + \partial/R\partial R$ (note that the

expression for $\Delta^\star$ given by Hopcraft in Dendy's textbook [10] is *not* the toroidal Laplacian and differs by the sign of the additional term), and the form of the additional geometric term hints at the solution. Direct integration yields the toroidal potential $\Phi_{tor} \equiv \Phi_{axi}(R \to \ln R)$, from which $E_Z = -\Phi_0[(\ln R - R_0)/2R_0^2 - 2/R_0]$ and $E_R = -\Phi_0 Z/2RR_0^2$, noting that the introduction of the logarithm breaks the usually obvious relation between the symbol for the magnitude of a quantity and the units associated with that quantity—carefully pulling the units into the leading coefficients of expressions ensures that they are respected. From these, we get the poloidal field

$$E_\theta = -\frac{1}{r}\frac{\partial \Phi}{\partial \theta} = \left(\frac{-\Phi_0}{r}\right)\left(\frac{\partial Z}{\partial \theta}\frac{\partial}{\partial Z} + \frac{\partial R}{\partial \theta}\frac{\partial}{\partial R}\right)\frac{\Phi}{\Phi_0} = E_\theta^{(2)}\left[(R-R_0)\frac{E_Z}{\Phi_0} - Z\frac{E_R}{\Phi_0}\right] \ , \tag{3}$$

where we identify $E_{r,\theta}^{(2)} \equiv -\Phi_0/r$, and the radial field

$$E_r = -\frac{\partial \Phi}{\partial r} = \left(\frac{-\Phi_0}{r}\right) r \left(\frac{\partial Z}{\partial r}\frac{\partial}{\partial Z} + \frac{\partial R}{\partial r}\frac{\partial}{\partial R}\right)\frac{\Phi}{\Phi_0} = E_r^{(2)}\left[-Z\frac{E_Z}{\Phi_0} - (R-R_0)\frac{E_R}{\Phi_0}\right] \ . \tag{4}$$

In (r, θ, ϕ) coordinates, we have $\Phi_{tor} = \Phi_0 r \sin\theta[-\ln(R_0 + r\cos\theta) + 5R_0]/2R_0^2$, and

$$E_\theta(r,\theta) = \frac{\Phi_0}{2rR_0^2}\left\{r\cos\theta\left[\ln\left(R_0 + r\cos\theta\right) - 5R_0\right] - \frac{r^2\sin^2\theta}{R_0 + r\cos\theta}\right\} \ , \tag{5}$$

$$E_r(r,\theta) = \frac{\Phi_0}{2rR_0^2}\left\{r\sin\theta\left[\ln\left(R_0 + r\cos\theta\right) - 5R_0\right] + \frac{r^2\cos\theta\sin\theta}{R_0 + r\cos\theta}\right\} \ . \tag{6}$$

As this is an electrostatic field supported by a neutral medium (*ie* one for which the net charge on a differential volume element vanishes), Maxwell's equations $\nabla \times \mathbf{E} = 0$ and $\nabla \cdot \mathbf{E} = 0$ are satisfied.

As electrostatic fields necessarily require a supporting charge density [3, 11, 12, 13, 14], one may very well ask where these charges are. As the solution to Laplace's equation is uniquely determined by the boundary condition, we might as well put them on the boundary of the region under consideration, which in this case is the R/R_0 weighted circle representing our outermost flux surface at normalized minor radius $r = 1$ upon collapse of the toroidal dimension, giving us an inverse Dirichlet problem, which is usually defined as solving for the potential given the boundary condition. As the current in a tokamak is observed in the toroidal direction, we suppose the plasma to behave as a dielectric in the plane orthogonal to the current, and take the polarization to be proportional to the electrostatic field, $\mathbf{P} \propto \mathbf{E}$, an admittedly crude approximation but one which satisfies the requirement of no volumetric bound charge density, $\rho_b = -\nabla \cdot \mathbf{P} = 0$. Were the plasma conductive in the $(Z, R) \cong (r, \theta)$

plane, no electrostatic field could be supported [3]. To find the surface bound charge density, we take $\sigma_b = \mathbf{P} \cdot \hat{r}$, noting that quantity represents the charge density on the weighted circle, so to get the physical charge density on a poloidal circle of the toroidal flux surface we take $\sigma_b \to \sigma_b R_0/R$. We note that usually it is an applied electric field which drives the polarization of dielectrics, but for a magnetized plasma other effects may support the polarization.

From the expression for the poloidal field, one may determine a prediction for the associated electron temperature profile by solving the flux surface Fourier moments of the poloidal equation of motion [5]. With electron density $n_e = n_e^0(1 + n_e^c \cos\theta + n_e^s \sin\theta)$ and temperature $kT_e \equiv Ŧ_e$, and the unity, cosine, and sine moments defined by the expressions $\langle A \rangle_{U,C,S} \equiv \oint d\theta \{1, \cos\theta, \sin\theta\}(1 + \varepsilon \cos\theta)A/2\pi$, we solve $\langle Ŧ_e \partial n_e / r\partial\theta + n_e e E_\theta \rangle_{U,C,S} = 0$ for (the pure numbers) n_e^c, n_e^s, and $E_\theta^{(2)}$, given by

$$\begin{bmatrix} n_e^c \\ n_e^s \\ E_\theta^{(2)} \end{bmatrix} = \begin{bmatrix} 3\varepsilon^3/G_2 \\ \pm\sqrt{6}\varepsilon G_1/2(F+2)G_2 \\ \pm 8\sqrt{6}Ŧ_e/e\varepsilon G_1 \end{bmatrix}, \tag{7}$$

where $F \equiv 5R_0 - \ln R_0$, in terms of the functions $G_2(r) \equiv [(6F-1)\varepsilon^2 - 8F]$ and

$$G_1(r) \equiv \sqrt{(F+2)[\{16F[4F - (1+3F)\varepsilon^2]\} + 9\varepsilon^4]}, \tag{8}$$

from which we obtain $\Phi_0 = \mp 8\sqrt{6} R_0 Ŧ_e / eG_1$. Note that the quantities appearing in, between, and following Equations (7) and (8) carry no units, as the units cancel out of the system of equations solved. The factor of R_0 appearing in the scale factor of Φ_0 has a physical interpretation—for an equivalent supporting charge density and minor radius, as one scales the system by R_0, the magnitude of the charge distribution on the weighted circle increases as the major circumference of the torus, thus increasing the magnitude of the potential by a numerical factor of R_0. As Φ_0 is constant $\forall\, r$, the explicit $r \equiv \varepsilon R_0$ dependence of G_1 must cancel the implicit r dependence of $Ŧ_e(r)$, allowing us to write $Ŧ_e(r)/G_1(r) \equiv Ŧ_e(0)/G_1(0)$, thus G_1 determines the r dependence of the electron temperature as a prediction of the hypothesis that the electrostatic field in a tokamak is of a form similar to that of Equation (2).

III. PRESENTATION

The following presents the results of a numerical evaluation of the expressions in the preceding section. The scale is set by $R_0 = 2$ and $Ŧ_e(0)/e = 1$, and r is the normalized

minor radius. Some useful geometry is presented in the Appendix. The toroidal electrostatic potential, Φ_{tor}, is shown in Figure 1—note the correct harmonic form of a stretched circular membrane (*eg* a drumhead put on a hoop given by the boundary condition), consistent with the physics of potential theory [6, 7], and the obvious lack of bumps or poles, consistent with a neutral medium (the term "pole" actually means something physical [3, 7, 15], namely the effect on the potential of an isolated flux-source which ultimately is quantized in units of the charge). We stress that these are multiple views of the same single object. In Figure 2 we display the magnitude of the associated electrostatic field, and in Figure 3 we display the electrostatic field and the associated supporting charge density.

The normalized electron temperature profile in a tokamak often compares favorably (from unpublished results of the analysis behind Reference [4]) with the formula $\mathcal{T}_e(r)/\mathcal{T}_e(0) \approx (1-r^4)^2$, which misses the effect of the pedestal but works surprisingly well for $r < .8$, and so that is the assumed profile to which we will make our comparison in Figure 4. The normalized predicted profile for the temperature is $G_1(r)/G_1(0)$ and fails to fall sufficiently quickly to match the shape of the expected temperature profile. Thus, while comparison with our approximate formula hardly qualifies as a true experimental test of the hypothesis [16], the existence of an electrostatic field of form similar to that of Equation (2) is not supported by this analysis.

IV. CONCLUSION

By starting with the neoclassical prediction for the poloidal electrostatic field in a tokamak, we have determined the corresponding electrostatic field appropriate for toroidal devices, as well as the supporting charge density on the boundary. Examination of the solution to the electron poloidal equation of motion reveals a prediction for the temperature profile, which does not compare well to that expected to be measured in a tokamak.

APPENDIX: SOME USEFUL GEOMETRY

There are (at least) three useful sets of coordinates to describe a toroidal magnetic confinement device with concentric circular flux surfaces, namely (Z, R, ϕ), (r, θ, ϕ), and $(r, \perp, \|)$, and in the infinite aspect ratio limit $(R_0 \to \infty)$ we have the axial coordinates

(Z, R, z) and (r, θ, z). (Note that the term "toroidal coordinates" means something very different to a mathematician than those commonly applied to a tokamak, which we call "tokamak coordinates.") The relationship between vectors in the tokamak coordinates $(Z, R, \phi) \leftarrow (r, \theta, \phi) \leftarrow (r, \perp, \|)$ may be succinctly expressed by

$$\begin{bmatrix} F_Z \\ F_R \\ F_\phi \end{bmatrix} = \begin{bmatrix} -\sin\theta & -\cos\theta & 0 \\ \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & b_\phi & b_\theta \\ 0 & -b_\theta & b_\phi \end{bmatrix} \begin{bmatrix} F_r \\ F_\perp \\ F_\| \end{bmatrix} , \tag{A.1}$$

where $\hat{\|} \equiv \mathbf{B}/B \equiv \hat{b} = (0, b_\theta, b_\phi)$. Various operators in tokamak coordinates are given by

$$\hat{Z}\frac{\partial f}{\partial Z} + \hat{R}\frac{\partial f}{\partial R} + \hat{\phi}\frac{1}{R}\frac{\partial f}{\partial \phi} \equiv \nabla f \equiv \hat{r}\frac{\partial f}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{\phi}\frac{1}{R}\frac{\partial f}{\partial \phi} , \tag{A.2}$$

$$\frac{\partial F_Z}{\partial Z} + \frac{1}{R}\frac{\partial RF_R}{\partial R} + \frac{1}{R}\frac{\partial F_\phi}{\partial \phi} \equiv \nabla\cdot\mathbf{F} \equiv \frac{1}{rR}\frac{\partial rRF_r}{\partial r} + \frac{1}{rR}\frac{\partial RF_\theta}{\partial \theta} + \frac{1}{R}\frac{\partial F_\phi}{\partial \phi} , \tag{A.3}$$

$$\begin{bmatrix} \frac{1}{R}\frac{\partial RF_\phi}{\partial R} - \frac{1}{R}\frac{\partial F_R}{\partial \phi} \\ \frac{1}{R}\frac{\partial F_Z}{\partial \phi} - \frac{\partial F_\phi}{\partial Z} \\ \frac{\partial F_R}{\partial Z} - \frac{\partial F_Z}{\partial R} \end{bmatrix} \equiv \nabla\times\mathbf{F} \equiv \begin{bmatrix} \frac{1}{rR}\frac{\partial RF_\phi}{\partial \theta} - \frac{1}{rR}\frac{\partial rF_\theta}{\partial \phi} \\ \frac{1}{R}\frac{\partial F_r}{\partial \phi} - \frac{1}{R}\frac{\partial RF_\phi}{\partial r} \\ \frac{1}{r}\frac{\partial rF_\theta}{\partial r} - \frac{1}{r}\frac{\partial F_r}{\partial \theta} \end{bmatrix} , \tag{A.4}$$

$$\frac{\partial^2 f}{\partial Z^2} + \frac{1}{R}\frac{\partial}{\partial R}R\frac{\partial f}{\partial R} + \frac{1}{R^2}\frac{\partial^2 f}{\partial \phi^2} \equiv \nabla^2 f \equiv \frac{1}{rR}\frac{\partial}{\partial r}rR\frac{\partial f}{\partial r} + \frac{1}{r^2R}\frac{\partial}{\partial \theta}R\frac{\partial f}{\partial \theta} + \frac{1}{R^2}\frac{\partial^2 f}{\partial \phi^2} , \tag{A.5}$$

and some useful relationships are

$$\begin{aligned} &Z = -r\sin\theta , && \partial Z/\partial r = Z/r , && \partial Z/\partial\theta = -(R - R_0) , \\ &R - R_0 = r\cos\theta , && \partial R/\partial r = (R - R_0)/r , && \partial R/\partial\theta = Z , \end{aligned} \tag{A.6}$$

and for the logarithm, we have $\partial \ln R/\partial R(m) = 1/R(m)$, which shows that the $\ln R$ carries no units.

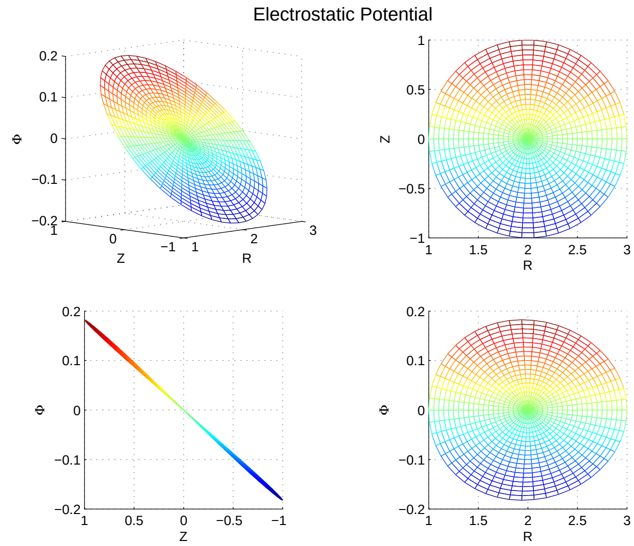

FIG. 1: (Color online.) The toroidal electrostatic potential Φ_{tor} seen from multiple views.

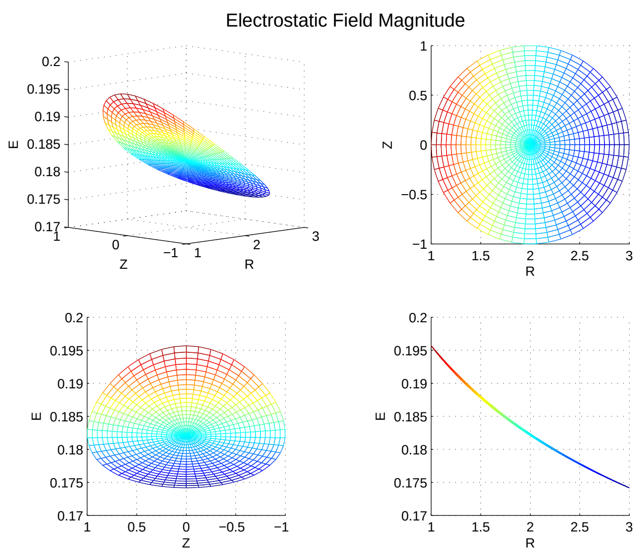

FIG. 2: (Color online.) The magnitude of the associated electrostatic field seen from multiple views.

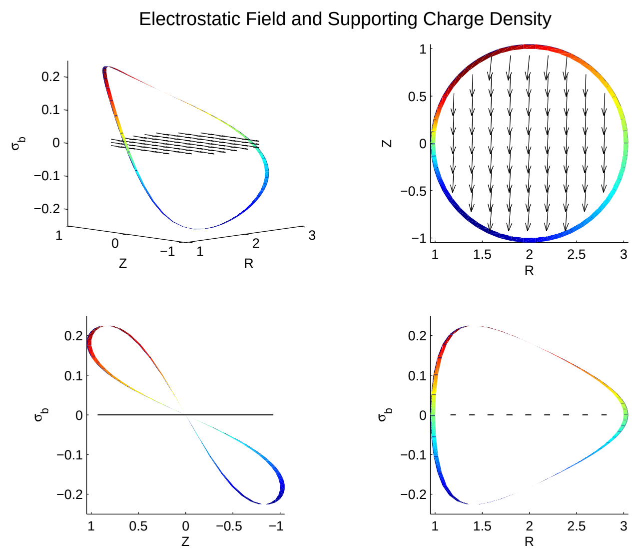

FIG. 3: (Color online.) The electrostatic field and associated supporting charge density seen from multiple views.

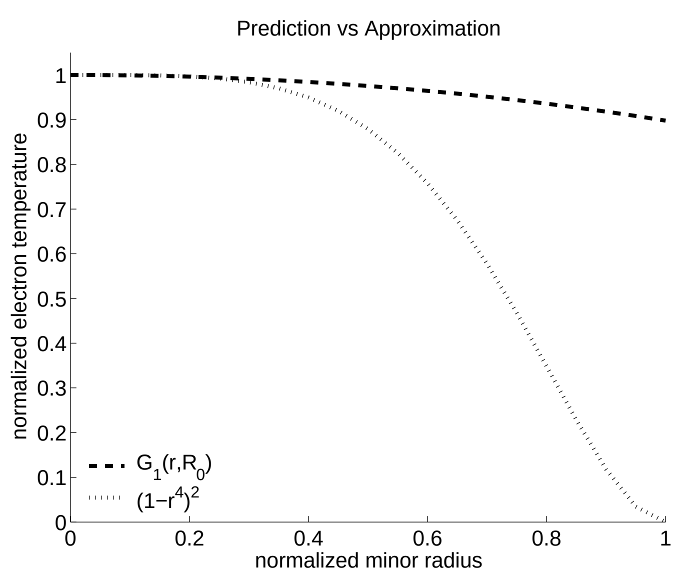

FIG. 4: Comparison of the neoclassical prediction for the normalized electron temperature profile, G_1, versus an approximation to that found in tokamak experiments, $(1-r^4)^2$.

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